Discontinuous Absorbing State Transition in $(1+1)$ Dimension

Urna Basu, Mahashweta Basu and P. K. Mohanty*

Theoretical Condensed Matter Physics Division, Saha Institute of Nuclear Physics, 1/AF Bidhan Nagar, Kolkata, 700064 India.

A $(1 + 1)$ dimensional model of directed percolation is introduced where sites on a tilted square lattice are connected to their neighbours by N channels, operated at both ends by valves which are either open or closed. The spreading fluid is assumed to propagate from any site to the neighbours in a specified direction only through those channels which have open valves at both sites. We show that the system undergoes a discontinuous absorbing state transition in the large N limit when the number of open valves at each site n crosses a threshold value $n_c = \sqrt{N}$. Remarkable dynamical properties of discontinuous transitions, like hysteresis and existence of two well separated fluctuating phases near the critical point are also observed. The transition is found to be discontinuous in all $(d + 1)$ dimensions.

Models of directed percolation (DP)[1] mimics filtering of fluids through porous media along a given direction. These models show a continuous phase transition from a non-percolating (absorbing) to a percolating (active) state when connectivity of pores is increased. Numerous physical phenomena, like forest fire[2], epidemics[3], transport in random media[4], synchronization[5] and chromatography can be understood using these models. The corresponding critical behaviour forms a universality class of absorbing state phase transitions, formally known as the DP-class, which has been realized [6]convincingly in (2+1) dimension.

In (1+1) dimension, the directed bond percolation[7] is modelled as a spreading process on a tilted square lattice where bonds connect any two neighbouring sites with probability p. A bond represents an open channel connecting two pores. On this random structure, the spreading agent (or the fluid) percolates along a given direction through the bonds (open channels). The system is said to be percolating if the agent spreads across an infinite cluster of connected bonds with finite probability. It is well known that percolation occurs here as a continuous (non-equilibrium) absorbing state phase transition when the microscopic connection probability p crosses a threshold value $p_c = 0.6447$[8]. In this article, we show that, *instead*, if sites are connected by multiple channels, the percolation can occur *discontinuously* by tuning relevant parameters.

There has been a long debate on the possibility of *discontinuous* absorbing state transition in one dimensional systems. It is conjectured by Hinrichsen[9] that absorbing state transition can not occur discontinuously in one dimensional systems under generic circumstances as strong fluctuations usually destabilize the ordered phase. This hypothesis challenged the existence of the discontinuous transition in one dimensional triplet creation model observed by Dickman and Tomé[10], and further supported by Cardoso and Fontanari[11]. Later studies show
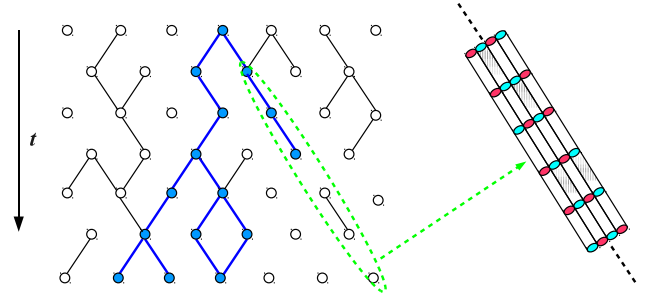

FIG. 1: (Color online) Illustration of bond directed percolation in $(1 + 1)$ dimension with $N = 4$ channels. Each site on a tilted square lattice has $N = 4$ valves(discs), of which $n = 2$ are open (blue) and the rest are closed(red). Existence of at least one channel which has open valves at both ends (shaded), represents a bond (solid lines) between the corresponding sites. Activity can percolate only through a bond along a preferred direction (time) activating new sites (filled circles). These bonds are shown as thick blue lines.

that the this transition is actually continuous[12, 13]. Again, the apparent discontinuous transition in annihilation/fission process[14], and the spreading process on a diffusing background[15], turned out[9] to be transient phenomena, crossing over to a continuous transition after very long time. Genuine 1st order absorbing state transitions are either seen in systems with special symmetries [16] or in higher dimensions[17, 18].

In this Letter we introduce a multichannel model of bond directed percolation in (d+1) dimensions where every pair of neighbouring sites has N different channels joining them. Each channel contains a valve at every site that can either be open or closed independent of other valves. A bond is said to connect two sites if there exists at least one channel joining them, which has open valves at both the ends. Like the usual model of bond directed percolation, the spreading agent here propagates in a specific direction starting from a random site. We show that the system undergoes a discontinuous percolation transition in the large N limit, even in $(1 + 1)$ dimension,

when the number of open valves n at each site crosses a threshold value $\sqrt{N}$. We believe that these results convincingly establish the existence of first order absorbing state transition in $(1+1)$ dimension.

First let us discuss this multichannel model of directed percolation in $(1+1)$ dimension in details. Its extension to higher dimension is straightforward. The sites on a tilted square lattice are connected to their neighbours by N directed channels (Fig. 1). Each channel contains a valve at every lattice site so that, at every site, there are N valves in total. Each valve can either be open or closed independent of other valves. Flow through a channel is possible only when the channel is *open*, *i.e.*, it has open valves at both ends. In analogy with the usual bond percolation process, a *bond* is said to be present between two neighbouring sites when there exists at least one open channel connecting them. The bond is absent if none of the connecting channels are open. At each site, randomly chosen n valves out of N are opened, which mimics a porous medium with randomly distributed directed *bonds* on $(1+1)$ dimensional tilted square lattice. Like the usual bond directed percolation problem, percolation is said to occur here if there exists a directed path which spans the system.

DP is often regarded as a dynamical spreading process where the fluid percolates in a given direction, interpreted as time. A similar interpretation for this multichannel model follows. In $(1+1)$ dimension, the lattice sites in horizontal and vertical directions are denoted by spatial and time coordinates $i = 1, 2 \ldots L$ and t respectively (Fig. 1). From any arbitrary site (i, t) the spreading fluid can independently propagate to the nearest neighbouring sites $(i \pm 1, t + 1)$ only if the target site is connected to (i, t) by a bond. In other words, the fluid can reach (activate) $(i \pm 1)^{th}$ site at time $t + 1$ from (i, t) if there is at least one open channel between $(i \pm 1, t + 1)$ and (i, t). If none of the sites at $t + 1$ gets activated, *i.e.* if the number of *active sites* at time $t + 1$ is zero, which happens when there is no bond connecting (i, t) and $(i \pm 1, t + 1)$ for any i, the spreading agent can not propagate further. Corresponding configuration at $t + 1$ is called *absorbing*. In this dynamical description, the system is said to be percolating when the fluid keeps spreading for arbitrarily long time. Note that when the system is percolating, there must exist one or more directed paths spanning the $(1 + 1)$ dimensional lattice. Thus, this dynamic description is completely equivalent with the geometric interpretation of the model discussed in the previous paragraph. Henceforth we continue with the dynamical description.

A continuous phase transition from the absorbing to active state occurs in the usual models of bond directed percolation when the connection probability p is increased beyond a critical threshold p_c. The steady state density of active sites ρ_s plays the role of the order parameter as it takes non-zero value only in the active phase $p > p_c$. In the multichannel model introduced here n plays the role of the control parameter as the connectivity between neighbouring sites increases when more
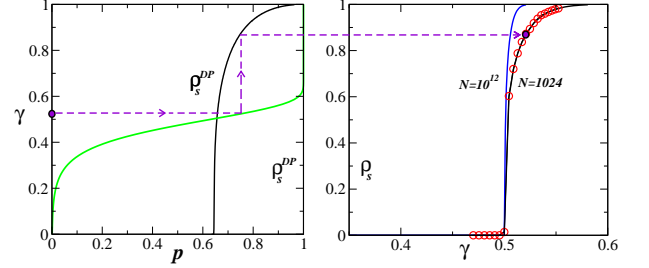

FIG. 2: (Color online) (a) The steady state density of active sites ρ_s^{DP} of usual bond directed percolation on a tilted square lattice with $L = 1000$ is plotted against p, along with γ versus p [calculated from Eq. (2)] for the multi channel model in $(1+1)$ D with $N = 1024$. For any value of γ, which corresponds to a unique p (thus ρ_s^{DP}) one can read out ρ_s using (4) [following the arrow]. The resulting ρ_s for $N = 1024$ and 10^{12} are shown as solid lines in (b). Symbols therein correspond to ρ_s obtained from direct numerical simulation of our model in $(1+1)$ D with $L = 1000$, $N = 1024$.

valves are open. Clearly there can not be any open channel for $n = 0$, hence there are no bonds available for spreading. Corresponding steady state density of active sites $\rho_s = 0$. Again, for $n > N/2$ every pair of neighbouring sites has at least one open channel connecting them. Starting from any random site the spreading agent surely activates all the sites in the steady state, resulting in $\rho_s = 1$. Our aim here is to find if the system percolates for any non-zero $n_c < N/2$.

Undirected bond percolation on similar multichannel models [19] are known to exhibit percolation transition in one and higher dimensions when the number of channels $N \to \infty$. The transition occurs discontinuously as the number of open valves n crosses $n_c = \sqrt{N}$. In the present work we investigate the possibility of an absorbing state transition by varying n along with N as $n = N^{\gamma}$ where $0 < \gamma < 1$.

Since at every lattice site, n valves are chosen randomly from the total N and opened, which can be done in C_n^N possible ways, the probability that there is no common open valve between any two neighbouring sites, *i.e.* the probability that neighbouring sites are *not* connected by any open channel, is $q = C_n^{N-n}/C_n^N$. Using Stirling's approximation, q can be expressed in terms of $\nu = n/N$ and N,

$$q = \frac{(1 - \nu)}{\sqrt{1 - 2\nu}} \exp\left[-g(\nu)N\right] \tag{1}$$

$$\text{where } g(\nu) = -\ln\left[(1 - \nu)\left(\frac{1 - \nu}{1 - 2\nu}\right)^{(1-2\nu)}\right].$$

Note that, here $p = 1 - q$ is the probability that neighbouring sites are connected by a bond. In the large N limit, $\nu = N^{\gamma-1}$ vanishes and $g(\nu) = \nu^2$ to the leading order in ν. This results in

$$p = 1 - \exp(-N^{2\gamma-1}). \tag{2}$$

Thus,

$$\lim_{N\to\infty} p = \begin{cases} 0 & \text{for } \gamma < 1/2 \\ 1 - e^{-1} & \text{for } \gamma = 1/2 \\ 1 & \text{for } \gamma > 1/2. \end{cases} \quad (3)$$

Clearly, the connection probability p shows a discontinuity at $\gamma_c = \frac{1}{2}$ in the $N \to \infty$ limit yielding a transition from the absorbing state (where there are no bonds) to an active phase (where all neighbouring sites are connected by bonds). Thus for any $\epsilon = (\gamma - \gamma_c) > 0$ the spreading agent eventually activates all the sites in steady state resulting in $\rho_s = 1$. This absorbing state transition is discontinuous as the order parameter ρ_s changes discontinuously from 0 to 1 as γ crosses $\gamma_c = \frac{1}{2}$.

The order parameter $\rho_s(\gamma; N)$ for any finite N can be calculated from the knowledge of the steady state density of active sites $\rho_s^{DP}(p)$ of usual bond directed percolation model for a given connection probability p,

$$\rho_s(\gamma; N) = \rho_s^{DP}(p = 1 - N^{2\gamma-1}). \quad (4)$$

As the analytical form of $\rho_s^{DP}(p)$ is not known, we first obtain the same numerically by simulating the usual bond directed percolation problem on a tilted square lattice with $L = 1000$ for $0 \le p \le 1$. $\rho_s(\gamma; N)$ can be read out from this data using Eq. (4) where $p = 1 - N^{2\gamma-1}$. is used. This procedure is illustrated in Fig. 2(a) for $N = 1024$. The resulting $\rho_s(\gamma; N)$ is shown in Fig. 2(b) as a solid line. The symbols therein correspond to the same obtained from direct simulation of the multi channel system on a tilted square lattice with $L = 1000$ having $N = 1024$ channels. It is clear from this figure that the transition occurs discontinuously near $\gamma_c = 1/2$. The order parameter $\rho_s(\gamma; N)$ for $N = 10^{12}$, obtained using the above procedure, is also shown in Fig. 2(b) to illustrate that, as expected, the transition point shifts towards $\gamma_c = 1/2$ as $N \to \infty$.

The existence of a discontinuity in the order parameter ρ_s is clear from the above discussions. However the 1st order phase transitions possess several remarkable dynamical aspects including hysteresis which distinguish them from the continuous ones. In the following we will show the numerical evidence of hysteresis, the most important signature of a discontinuous transition. Before going into the details of these studies first we check that the time-series of the instantaneous density of active sites $\rho(t)$ shows a sharp drop when the tuning parameter γ is suddenly decreased below the critical threshold γ_c[marked as $t = 0$ in Fig. 3(a)]. Such a finite drop does not occur in the models of usual directed percolation as the order parameter there vanishes continuously at p_c.

In any discontinuous transition, it is expected that the system near the critical threshold fluctuates between the ordered and the disordered phases leading to interesting dynamical behaviour. However it is difficult to study these signatures in a discontinuous absorbing state transition, as once a system reaches an absorbing configuration it remains there forever. To overcome this difficulty,
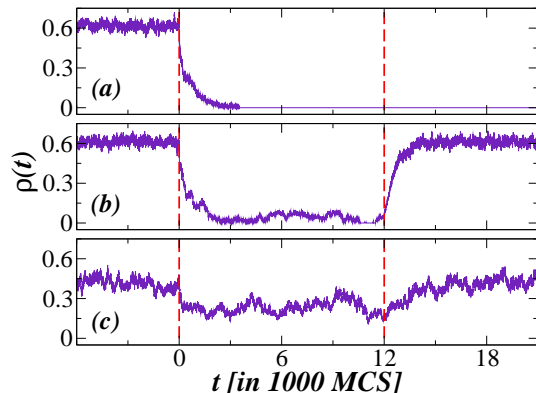

FIG. 3: (Color online)$\rho(t)$ near the transition point: The system in steady state with $\gamma = \gamma_c + \epsilon_1$ is quenched to $\gamma = \gamma_c - \epsilon_2$ at $t = 0$ and the γ is changed back to its initial value after 12000 MCS. (a) and (b) correspond to our model with $\delta = 0, 10^{-5}$ respectively. (c) shows $\rho(t)$ for conventional DP near the critical point p_c with same ϵ_1, ϵ_2 and δ. Here $\epsilon_1 = 0.005$ and $\epsilon_2 = 0.001$.

Bidaux et. al. [17] have introduced a special technique where inactive sites are spontaneously activated with a small rate δ. This rate δ does not change the discontinuous nature of the transition but converts the absorbing state into a fluctuating low density state allowing one to study hysteresis and related dynamical aspects of the discontinuous absorbing state transition. In presence of δ, the transition in this system occurs from a high density to a low density phase. Figure 3(b) shows the time series of $\rho(t)$, when the system in steady state at $\gamma = \gamma_c + \epsilon_1 = 0.505$ ($\epsilon_1 = 0.005$) is quenched to $\gamma = \gamma_c - \epsilon_2 = 0.499$ ($\epsilon_2 = 0.001$) and kept there for 12000 Monte Carlo steps (MCS). In the Monte-Carlo simulation we have used $\delta = 10^{-5}$. Here, $\rho(t)$ drops to the low density state soon after the quench ($t = 0$) and again jumps back to the high density phase as γ reverts back to the initial value. This behaviour is typical of 1st order phase transitions. A similar variation of the tuning parameter across the critical point of ordinary directed percolation does not show [Fig. 3(c)] any visible change in $\rho(t)$. This is because of the fact that in absence of the absorbing state, the steady state density does not change appreciably across p_c.

Hysteresis: To study hysteresis the system is first relaxed at $n > \sqrt{N}$ such that the corresponding γ is well above the critical threshold. Then n is decreased in steps of 1 after every t_R Monte-Carlo steps (MCS) until a sufficiently low value of γ below γ_c is reached. Now the path is reversed *i.e.* n is increased in steps of 1, and the hysteresis cycle is completed. In Fig. 4, we have shown the hysteresis curves in the γ-ρ plot for a system of $L = 1000, N = 1024$ with different $t_R = 10, 30$. The fact that the hysteresis curve bounds a finite area reinforces the claim that this absorbing state transition is discontinuous. As expected this area decreases with the increase of t_R.
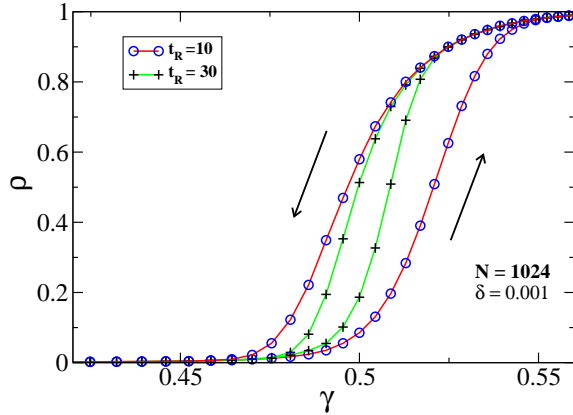
FIG. 4: (Color online) The multichannel model shows hysteresis near the transition point. Here activity is created spontaneously with a small rate $\delta = 10^{-5}$. In the Monte-Carlo simulation we have used $L = 1000$, $N = 1024$. The two loops correspond to the relaxation times $t_R = 10$ (circle) and $t_R = 30$ (plus). Arrows indicate the direction in which the loops are generated.

Site DP: The model can be recast into a different form by associating a set S of n integers, randomly chosen from a larger set $\{1, 2, \ldots N\}$, to each lattice site. In this picture, a bond is said to connect two neighbouring sites (i, t) and $(i \pm 1, t+1)$ if $S(i, t) \cap S(i \pm 1, t+1) \neq \emptyset$, where $\emptyset$ is the null set (an empty set without any element). The larger set, here, corresponds to N valves existing at each site and the smaller set S corresponds to the n open valves. In fact, this picture is useful in describing site directed percolation in this multichannel model.

In case of site directed percolation, each site i at time t contains a set $S(i, t)$ which is either a null set $\emptyset$ (inactive) or a set of n integers randomly chosen from $\{1, 2, \ldots N\}$. A site is said to be active when the associated set is non-empty ($S \neq \emptyset$). From any given configuration at t the next configuration at time $t + 1$ can be generated as follows. If both $S(i \pm 1, t) = \emptyset$, then $S(i, t+1)$ is assigned the null set $\emptyset$. Otherwise, *i.e.* when at least one of them is not $\emptyset$, a trial set T of n integers is created and $S(i, t+1)$ is updated as,

$$S(i, t+1) = \begin{cases} T & \text{if } T \cap K \neq \emptyset \\ \emptyset & \text{if } T \cap K = \emptyset. \end{cases} \quad (5)$$

where K is one of $S(i \pm 1, t)$ if both are non-empty; otherwise K is the only non-empty set among $S(i \pm 1, t)$.

Evidently, through this dynamics, new active sites are created only from the existing neighbouring active sites with probability p which follow Eq. (2) when n varies as $n = N^\gamma$. Thus, in $N \to \infty$ limit, here too one expects a discontinuous transition similar to that of the bond percolation model. Details of this study are omitted here.

It is straight forward to study this multichannel model of directed percolation on any kind of lattice in any $(d + 1)$ dimension. In higher dimensions also, one expects the transition to occur discontinuously at $\gamma_c = \frac{1}{2}$ (*i.e.* $n_c = \sqrt{N}$) as the connection probability p [Eq. (2)] itself becomes discontinuous when number of channels $N \to \infty$.

In summary, we have studied directed percolation in $(d+1)$ dimension on a multi channel system where neighbouring sites are connected by N channels, each having an operating valve at either ends. Of these N valves at each site, n are open. The fluid can percolate in a given direction, only through the channels which have open valves at both ends. In the $N \to \infty$ limit, this system shows a discontinuous transition from a percolating phase to a non-percolating (absorbing) state as n is decreased below a threshold $n_c = \sqrt{N}$.

It is rather surprising that the absorbing state transition in this $(1 + 1)$ dimensional model is discontinuous. The possibility of such a transition has been rebutted by Hinrichsen [9], who argued that unless there is an unusually robust mechanism to prevent fluctuation from destabilizing the ordered phase, it is impossible to have a discontinuous absorbing state transition in generic $(1+1)$ dimensional systems with short range interactions. However, this argument does not seem to apply here, as the discontinuity in the order parameter is the artefact of the discontinuity that appears in the resulting connection probability while varying the control parameter γ continuously across $\gamma_c = 1/2$; the role of fluctuation is immaterial here.

It is worth mentioning that depinning or synchronization, which are described by a multiplicative noise Langevin equation[20] similar to that of DP, show discontinuous transitions in certain model systems [21]. It would be interesting to investigate whether the discontinuous absorbing state transition obtained in the present study falls in the realm of their theory.

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